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Structure of matrix elements in the quantum Toda chain

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Abstract. We consider the quantum Toda chain using the method of separation of variables. We show that the matrix elements of operators in the model are written in terms of a finite number of 'deformed Abelian integrals'. The properties of these integrals are discussed. We explain that these properties are necessary in order to provide the correct number of independent operators. A comparison with the classical theory is made.

1. Introduction

As became clear recently [1–3] there is a close connection between the formula for the matrix elements in the integrable field theory (form factors) [4] and the method of separation of variables developed by Sklyanin [5].

The form factors are typically given by certain integrals. This kind of formula can be interpreted as follows. Consider an integrable model which allows the separation of variables. The separated variables naturally split into two equal parts: one of them can be considered as 'coordinates' and another as 'momenta' (of course they have nothing to do with the original canonical variables in which the model is formulated). The formulae for the form factors are understood as matrix elements written in 'coordinate' representation, i.e. in terms of integrals with respect to the 'coordinates'.

Another observation made in [6], and used intensively in [1, 2] is that the integrals in the formulae for the form factors in models with $\widehat{sl}(2)$ Lie–Poisson symmetry (sine-Gordon, for example) can be considered as deformations of hyperelliptic integrals. This fact must also be related to the method of separation of variables because the 'coordinates' describe classically a divisor on the spectral hyperelliptic curve. The important conclusion made in [2] is that these deformed hyperelliptic integrals must have similar properties to the usual hyperelliptic integral in order that the correct number of equations of motion exists in the quantum case.

In paper [3] we performed the quasiclassical analysis of the matrix elements in the conformal field theory (CFT) in finite volume. This is a much more complicated case than the case of infinite volume. The method of separation of variables seems to give the only possible approach to the calculation of form factors. The main difficulty of the problem in finite volume is due to the fact that the separation of variables leads to the Baxter equations whose solutions describe the wavefunctions in the 'coordinate' representation. So, one must consider integrals over solutions of Baxter equations.

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In this paper we consider a much simpler model which nevertheless exhibits difficulties similar to those of integrable field theory in finite volume. This is the periodical Toda chain. Historically this is the first model to which the method of separation of variables was applied [5]. In this case the problem of describing the spectrum leads to Baxter equations with nontrivial solutions in entire functions. The matrix elements are given by integrals over these solutions. We show that these integrals can be considered as deformed hyperelliptic integrals allowing a deformation of all the important properties of hyperelliptic integrals. Similarly to [2] these properties are needed for the correct counting of operators: they are actually equivalent to the equations of motion.

2. Classical Toda chain

The periodical Toda chain is described by the Hamiltonian:

$$H = \sum_{j=1}^{n} \frac{p_j^2}{2} + \mathrm{e}^{q_{j+1}-q_j} \tag{1}$$

where p_j, q_j are canonical variables, $q_{n+1} \equiv q_1$.

The exact solution is due to existence of a Lax representation. Consider the L-operator

$$L_j(\lambda) = \begin{pmatrix} \lambda - p_j, & \mathrm{e}^{q_j} \\ & \\ -\mathrm{e}^{-q_j}, & 0 \end{pmatrix}$$

and the monodromy matrix

$$M(\lambda) = L_n(\lambda) \ldots L_1(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}.$$

Obviously, $\det M(\lambda) = 1$. The monodromy matrix satisfies Sklyanin's Poisson brackets:

$$\{M(\lambda) \overset{\otimes}{,} M(\mu)\} = [r(\lambda - \mu), M(\lambda) \otimes M(\mu)] \tag{2}$$

where

$$r(\lambda) = -\frac{P}{\lambda}.$$

P is the permutation. The coefficients of $T(\lambda) \equiv \operatorname{tr} M(\lambda)$ are in involution:

$$\{T(\lambda), T(\mu)\} = 0.$$

Moreover,

$$T(\lambda) = \lambda^n - P\lambda^{n-1} + (\tfrac{1}{2}P^2 - H)\lambda^{n-2} + \cdots$$

where $P = \sum p_j$ is the total momentum and H is the Hamiltonian (1). Thus $T(\lambda)$ generates n integrals of motion in involution providing complete integrability of the system.

From here on we can forget about the Toda chain saying that we consider an orbit of Lie–Poisson group [7] i.e. the polynomial matrix $M(\lambda)$ with $\det M(\lambda) = 1$ satisfying the Poisson brackets (2) (determinant is in the centre of these Poisson brackets) and characterized by certain reality conditions which we shall discuss later.

Let us consider the elements of $M(\lambda)$ in some more details. We introduce the notations

$$\begin{aligned} A(\lambda) &= \lambda^n + \lambda^{n-1}a_1 + \cdots + a_n \\ B(\lambda) &= b(\lambda^{n-1} + \lambda^{n-2}b_1 + \cdots + b_{n-1}) \\ C(\lambda) &= \lambda^{n-1}c_2 + \cdots + c_{n+1} \\ D(\lambda) &= \lambda^{n-2}d_2 + \cdots + d_n. \end{aligned}$$

The variables b and a_1 have the Poisson brackets

$$\{a_1, b\} = b$$

and Poisson-commute with the rest of the variables. In terms of the Toda chain $a_1 = P$ and $b = e^{q_n}$ describe the motion of the centre of mass. Our nearest concern is the algebra of observables $\mathcal{A}$. We define this algebra as the one generated by all the monomials of finite degree of the variables a_j, b_j, c_j, d_j and b. It is important that the polynomial structure of $M(\lambda)$ introduces grading of $\mathcal{A}$. Namely, we can prescribe the degree i to the variables a_i, b_i, c_i, d_i and the degree 0 to b. The degrees of the leading coefficients of $C(\lambda)$ and $D(\lambda)$ are chosen in order that the coefficients of the determinant

$$\det M(\lambda) = \lambda^{2n-2} f_2 + \cdots + f_{2n}$$

are homogeneous. The variable b is a kind of a zero-mode, it is of minor dynamical value. The algebra $\mathcal{A}$ contains a subalgebra $\mathcal{A}_0$ of polynomial functions of a_i, b_i, d_i and $\tilde{c}_i = bc_i$. So, this subalgebra does not have b as a separate generator, the change in the definition of c_i is needed in order that the Poisson brackets are closed for $A(\lambda)$, $D(\lambda)$, $\tilde{B}(\lambda) = \prod(\lambda - \gamma_j)$ and $\tilde{C}(\lambda) = bC(\lambda)$. We shall deal only with this subalgebra.

The algebra $\mathcal{A}_0$ as a vector space splits into a direct sum of subspaces of different degrees. Let us denote by $\delta(n)$ the dimension of the subspace of the degree n. The generating function of $\delta(n)$ (character) is given by

$$\chi(q) \equiv \sum_{n=0}^{\infty} \delta(n) q^n = \frac{1}{[n]!} \frac{1}{[n-1]!} \frac{[1]}{[n+1]!} \frac{[1]}{[n]!} \frac{[2n]!}{[1]}$$

where $[n] = 1 - q^n$, $[n]! = [1][2] \ldots [n]$. The first four multipliers come from monomials of $a_j, b_j, \tilde{c}_j, d_j$ respectively, the last multiplier comes from the factorization by the condition $\det M(\lambda) = 1$.

Notice that

$$\chi(q) = \frac{1}{[n]![n-1]!} \left(\begin{bmatrix} 2n-1 \\ n-1 \end{bmatrix} - q \begin{bmatrix} 2n-1 \\ n-2 \end{bmatrix} \right) \tag{3}$$

where we introduced the q-binomial coefficients

$$\begin{bmatrix} n \\ m \end{bmatrix} = \frac{[n]!}{[m]![n-m]!}.$$

Later we shall provide an interesting interpretation of this formula.

Let us return to a more traditional consideration of the classical Toda chain. We do not give a complete list of references, which can be found in [5]. For us the important fact concerning the classical system is that it allows the separation of variables [8, 9]. Consider zeros of the polynomial $B(\lambda)$:

$$B(\lambda) = b \prod_{j=1}^{n-1} (\lambda - \gamma_j)$$

and the variables $\Lambda_j \equiv D(\gamma_j)$. Notice that $\Lambda_j = \Lambda(\gamma_j)$ where $\Lambda(\lambda)$ is the eigenvalue of $M(\lambda)$. The variables γ_j, $\log \Lambda_j$ are canonocally conjugated which can be shown following [5] using (2):

$$\{\gamma_i, \log \Lambda_j\} = \delta_{i,j}.$$

From $\det M(\lambda) = 1$ it follows that $A(\gamma_j) = \Lambda_j^{-1}$. One can reconstruct the matrix $M(\lambda)$ from $\gamma_1, \ldots, \gamma_{n-1}$, $\Lambda_1, \ldots, \Lambda_{n-1}$, a_1 and b. The symplectic form is written as

$$\omega = \sum_{j=1}^{n-1} \mathrm{d}\log\, \Lambda_j \wedge \mathrm{d}\gamma_j + \mathrm{d}\log\, b \wedge \mathrm{d}a_1.$$

The 1-form α ($\omega = \mathrm{d}\alpha$) is

$$\alpha = \sum_{j=1}^{n-1} \log \Lambda_j\, \mathrm{d}\gamma_j + \log b\, \mathrm{d}a_1.$$

Let us take other coordinates on the phase space, namely, $\gamma_1, \ldots, \gamma_{n-1}$, $t_2, \ldots, t_n$ (defined by $T(\lambda) = \lambda^n + \lambda^{n-1}t_1 + \lambda^{n-2}t_2 + \cdots + t_n$), $t_1 \equiv a_1$ and b. From

$$\Lambda_j = \Lambda(\gamma_j) = \tfrac{1}{2}(T\left(\gamma_j) + \sqrt{T(\gamma_j)^2 - 4}\right)$$

one easily finds the expression for the symplectic form in these variables

$$\omega = \sum_{j=1}^{n-1} \sum_{k=2}^{n} \frac{\gamma_j^{n-k}}{\sqrt{P(\gamma_j)}} \mathrm{d}t_k \wedge \mathrm{d}\gamma_j + \mathrm{d}\log\, b \wedge \mathrm{d}a_1$$

where $P(\lambda) = T^2(\lambda) - 4$. Thus the equations of motion take the form

$$\begin{aligned} &\{T(\lambda), \gamma_j\} = \sqrt{P(\gamma_j)} \prod_{k\neq j} \frac{\lambda - \gamma_k}{\gamma_j - \gamma_k} \\ &\{T(\lambda), b\} = \lambda^{n-1} b. \end{aligned} \tag{4}$$

Only the first $n - 1$ equations are really interesting. They are linearized by the Abel transformation:

$$\left\{T(\lambda), \sum_{k=1}^{n-1} \int^{\gamma_k} \sigma_j\right\} = \lambda^{j-1}$$

where σ_j are first kind Abelian differentials on the spectral curve $\mu^2 = P(\lambda)$:

$$\sigma_j = \frac{\lambda^{j-1}}{\sqrt{P(\lambda)}} \mathrm{d}\lambda.$$

We associate the 'times' $\tau_1, \ldots, \tau_{n-1}$ with $t_2, \ldots, t_n$:

$$\frac{\partial}{\partial \tau_j} F \equiv \partial_j F = \{t_{j+1}, F\}.$$

The evolution of $\sum_{k=1}^{n-1} \int^{\gamma_k} \sigma_j$ with respect to times is linear.

The above considerations apply to any orbit of the Lie–Poisson group. We want now to consider specific reality conditions which correspond to the Toda chain. It can be shown [8, 9] that the conditions in question are:

(i) The polynomial $T(\lambda)$ of degree n has n real zeros. Moreover its local maxima are not below 2 and its local minima are not above -2. So, all the zeros of the polynomial $P(\lambda)$ are also real, they are denoted by $\lambda_1 < \lambda_2 < \cdots < \lambda_{2n}$.

(ii) The polynomial $B(\lambda)$ has real zeros $\gamma_1, \ldots, \gamma_{n-1}$ which belong to the 'forbidden zones': $\lambda_{2k} < \gamma_k < \lambda_{2k+1}$.

The equations of motion (4) preserve these conditions. The hyperelliptic Riemann surface $\mu^2 = P(\lambda)$ has $2n$ branch points (λ_j). Its genus equals $n-1$. We present the surface as two complex planes with the cuts along $(-\infty, \lambda_1]$, $[\lambda_2, \lambda_3], \ldots, [\lambda_{2n}, \infty)$ identifying the banks of the cuts on two sheets in usual way. The canonocal a-cycles a_j are taken as ones encircling the cuts $[\lambda_{2j}, \lambda_{2j+1}]$ for $j = 1, \ldots, n-1$. Topologically the points γ_j move along the cycles a_j.

Define the normalized holomorphic differentials

$$\omega_j = A_{jk}\sigma_k$$

such that

$$\frac{1}{2\pi}\int_{a_j}\omega_k = \delta_{j,k}.$$

Then

$$\theta_j = \sum_{k=1}^{n-1}\int^{\gamma_k}\omega_j$$

are real angles of the Jacobi variety, and the dynamics describes a linear motion along this real torus. One can invert the Abel transformation expressing the symmetric functions of $\gamma_1, \ldots, \gamma_{n-1}$ (recall that they coincide with $b_1, \ldots, b_{n-1}$) as functions of the Jacobi variety (functions of θ's) using the Riemann θ-function but we shall not need the explicit formulae. The angles θ and the times τ are related linearly:

$$\theta_j = \sum_{l=1}^{n-1} A_{jl}\tau_{n-l}$$

so, using the θ-function formulae mentioned above one can resolve the equations of motion expressing b_j as $b_j = b_j(\tau_1, \ldots, \tau_{n-1})$.

From the point of view of algebraic geometry the monodromy matrix $M(\lambda)$ gives an affine model of hyperelliptic Jacobian, and the functions $b_j(\tau)$ are the generalized Weierstrass functions [10]. In the case of genus one ($n = 2$) the function $\gamma(\tau_1) = b_1(\tau_1)$ is the usual Weierstrass function which satisfies the second-order differential equation

$$\partial_1^2\gamma = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\gamma}P(\gamma). \tag{5}$$

One of the results of our further analysis will be in finding certain second-order partial differential equations for generalized Weierstrass functions which can be thought of as generalizations of (5).

Let us consider the ring of generalized Weierstrass functions with coefficients in $t_1, \ldots, t_{n-1}$, i.e. the ring of polynomials

$$F(t_1, \ldots t_n, b_1, \ldots, b_{n-1}).$$

Consider further all possible derivatives of these polynomials with respect to τ_i:

$$\partial_1^{k_1}\ldots\partial_{n-1}^{k_{n-1}}F(t_1, \ldots, t_n, b_1, \ldots, b_{n-1}). \tag{6}$$

The equations we are looking for correspond to all possible linear combinations of the functions (6) which vanish due to equations of motion. To understand the origin of these equations we have to return to our mechanical considerations.

Mechanically one understands the derivatives ∂_i as Hamiltonian vector fields. Using the Poisson brackets (2) one can express (6) as a function of $a_1, \ldots, a_n$, $b_1, \ldots, b_{n-1}$, $\tilde{c}_2, \ldots, \tilde{c}_{n+1}$, $d_2, \ldots, d_n$ i.e. as an element of the algebra $\mathcal{A}_0$. We put forward the following.

Conjecture 1. Every element of $\mathcal{A}_0$ can be presented as a linear combination of the expressions (6).

We were not able to find a complete proof of this statement; however, the consideration of examples supports it. Further indirect support of this conjecture will be provided by the calculation of characters given below.

Assuming that the conjecture is true one realizes that the way of presenting an element of $\mathcal{A}_0$ as a linear combination of the expressions (6) may be not unique. Indeed, let us calculate the character of the space span by (6). We prescribe the degree i to ∂_i which is consistent with the Poisson brackets (2). Obviously, the character is

$$\frac{1}{[n-1]![n]![n-1]!} > \chi(q)$$

where $\chi(q)$ is the character (3). So, there must be a linear dependence between the functions (6) which is responsible for differential equations on the generalized Weierstrass functions. Moreover, there is a criterion which allows one to judge whether the set of equations is complete. Indeed, to show the completeness of the equations one has, obviously, to prove that taking them into account leads to the correct character (3).

Let us find the equations in question. To this end we shall use the Fourier transform. Consider a function $F(t_1,\dots t_n,b_1,\dots,b_{n-1})$. The variables t_j are the integrals of motion (and the moduli of the Riemann surface) and the variables b_j are the functions on the Jacobi variety due to the equations of motion. Hence

$$F(t_1,\dots t_n,b_1(\tau),\dots,b_{n-1}(\tau))\sum_{k_1,\cdots,k_{n-1}}\mathrm{e}^{-\mathrm{i}\Sigma k_j\theta_j}$$
$$\times\int_0^{2\pi}\mathrm{d}\theta_1'\dots\int_0^{2\pi}\mathrm{d}\theta_{n-1}'F(t_1,\dots,t_n,b_1(\theta'),\dots,b_{n-1}(\theta'))\mathrm{e}^{\mathrm{i}\Sigma k_j\theta_j'}$$

where $\theta_j=\sum_l A_{jl}\tau_{n-l}$. Let us undo the Abel transformation inside the integrals:

$$F(t_1,\dots,t_n,b_1(\tau),\dots,b_{n-1}(\tau))$$
$$eql\frac{1}{\det(A)}\sum_{k_1,\dots,k_{n-1}}\mathrm{e}^{-\mathrm{i}\Sigma k_j\theta_j}\int_{a_1}\frac{\mathrm{d}\gamma_1}{\sqrt{P(\gamma_1)}}\cdots\int_{a_{n-1}}\frac{\mathrm{d}\gamma_1}{\sqrt{P(\gamma_{n-1})}}\prod_{i<j}(\gamma_i-\gamma_j)$$
$$\times\tilde{F}(t_1,\dots,t_n,\gamma_1,\dots,\gamma_{n-1})\prod_j\mathrm{e}^{\mathrm{i}\Phi_k(\gamma_j)} \qquad (7)$$

where $\tilde{F}(t,\gamma)=F(t,b(\gamma))$ (recall that b_j are elementary symmetric functions of γ's). For any $k=\{k_1,\dots,k_{n-1}\}$ we define

$$\Phi_k(\gamma)=\int^{\gamma}k_j\omega_j.$$

Deriving (7) we needed to take into account the Jacobian of the Abel transformation. Later we shall relate the integrals in (7) to the quasiclassical limit of the matrix elements in the quantum Toda chain. The equations of motion correspond to vanishing of all the integrals in (7). Let us explain the possible reasons for these integrals to vanish.

Consider first the term in (7) with $k=0$ which is nothing but the average of F over the Jacobi variety (the trajectory):

$$\langle F\rangle=\frac{1}{\det(A)}\int_{a_1}\frac{\mathrm{d}\gamma_1}{\sqrt{P(\gamma_1)}}\cdots\int_{a_{n-1}}\frac{\mathrm{d}\gamma_1}{\sqrt{P(\gamma_{n-1})}}\prod_{i<j}(\gamma_i-\gamma_j)\tilde{F}(t,\gamma). \qquad (8)$$

There are two reasons for this integral to vanish. The first one is obvious, it is due to existance of exact forms. With an arbitrary polynomial $L(\gamma)$ associate the polynomial

$$D_t(L)(\gamma)=P(\gamma)\frac{\mathrm{d}L(\gamma)}{\mathrm{d}\gamma}+\frac{1}{2}\frac{\mathrm{d}P(\gamma)}{\mathrm{d}\gamma}L(\gamma).$$

We marked explicitly the dependence on the moduli (integrals of motion) $t = \{t_1, \ldots, t_n\}$ which enters through $P(\gamma)$.

There is an obvious proposition.

Proposition 1. The following integral vanishes

$$\int_c \frac{1}{\sqrt{P(\gamma)}} D_t(L)(\gamma) = 0 \tag{9}$$

for any polynomial L and any closed cycle c.

Hence the integal (8) vanishes if

$$\tilde{F}(t,\gamma) = \sum_{i=1}^{n-1} \frac{1}{\prod_{j\neq i}(\gamma_i - \gamma_j)} D_t(L)(\gamma_i) G(\gamma_1, \ldots, \widehat{\gamma_i} \ldots, \gamma_{n-1})$$

for any polynomial L and any symmetric polynomial of $n-2$ variables G (both of them can be also polynomials of parameters t). This property means in particular that by adding exact forms one can reduce the degree of the polynomial $\tilde{F}$ in every γ_j up to n.

The second reason for the integral (8) to vanish is due to the Riemann bilinear identity. Consider the antisymmetric polynomial of two variables

$$C_t(\gamma_1, \gamma_2) = R_t(\gamma_1, \gamma_2) - R_t(\gamma_2, \gamma_1) \tag{10}$$

where

$$R_t(\gamma_1, \gamma_2) = \sqrt{P(\gamma_1)} \frac{\mathrm{d}}{\mathrm{d}\gamma_1} \left(\frac{1}{\gamma_1 - \gamma_2} \sqrt{P(\gamma_1)} \right).$$

For any two cycles on the Riemann surface one has

$$\int_{c_1} \int_{c_2} \frac{1}{\sqrt{P(\gamma_1)}} \frac{1}{\sqrt{P(\gamma_1)}} C_t(\gamma_1, \gamma_2) = c_1 \circ c_2$$

where $\circ$ means the intersection number. Since the cycles a_j do not intersect one has the following proposition.

Proposition 2. For any two a-cycles a_j and a_k the following integral vanishes:

$$\int_{a_j} \int_{a_k} \frac{1}{\sqrt{P(\gamma_1)}} \frac{1}{\sqrt{P(\gamma_1)}} C_t(\gamma_1, \gamma_2) = 0.$$

Hence the integral (8) vanishes if

$$\tilde{F}(t,\gamma) = \sum_{i<j} \frac{1}{(\gamma_i - \gamma_j) \prod_{l\neq i,j}(\gamma_i - \gamma_l)(\gamma_j - \gamma_l)} C_t(\gamma_i, \gamma_j) G(\gamma_1, \ldots, \widehat{\gamma_i} \ldots, \widehat{\gamma_j} \ldots, \gamma_{n-1}).$$

Let us consider now the case of arbitrary $k = \{k_1, \ldots, k_{n-1}\}$. Introduce the polynomials $S_{t,k}$:

$$\mathrm{i} \sum_{j=i}^{n-1} k_j \omega_j(\gamma) = \frac{S_{t,k}(\gamma)}{\sqrt{P(\gamma)}} \, \mathrm{d}\gamma.$$

Integrating by parts one gets the following three simple propositions.

Proposition 1′. For any polynomial L define the polynomial

$$D_{t,k}(L)(\gamma) = D_t(L)(\gamma) - S_{t,k}(\gamma) \int_0^{\gamma} L(\gamma') S_{t,k}(\gamma') \, \mathrm{d}\gamma'$$

then the following integral vanishes for any a-cycle:

$$\int_{a_j} \frac{\mathrm{d}\gamma}{\sqrt{P(\gamma)}} \mathrm{e}^{\mathrm{i}\Phi_k(\gamma)} D_{t,k}(L)(\gamma) = 0.$$

Proposition 2′. Define the polynomial

$$C_{t,k}(\gamma_1, \gamma_2) = C_t(\gamma_1, \gamma_2) - S_k(\gamma_1) \int_0^{\gamma_1} \frac{S_{t,k}(\gamma) - S_{t,k}(\gamma_2)}{\gamma - \gamma_2} \,\mathrm{d}\gamma + S_k(\gamma_2) \int_0^{\gamma_2} \frac{S_{t,k}(\gamma) - S_{t,k}(\gamma_1)}{\gamma - \gamma_1} \partial\gamma$$

then for any two a-cycles a_j and a_k the following integral vanishes:

$$\int_{a_j} \frac{\mathrm{d}\gamma_1}{\sqrt{P(\gamma_1)}} \int_{a_k} \frac{\mathrm{d}\gamma_1}{\sqrt{P(\gamma_2)}} \mathrm{e}^{\mathrm{i}\Phi_k(\gamma_2)} \mathrm{e}^{\mathrm{i}\Phi_k(\gamma_1)} C_{t,k}(\gamma_1, \gamma_2) = 0.$$

There is one more identity which is trivial in the case $k = 0$.

Proposition 3′. For any a-cycle the following integral vanishes:

$$\int_{a_j} \frac{\mathrm{d}\gamma}{\sqrt{P(\gamma)}} \mathrm{e}^{\mathrm{i}\Phi_k(\gamma_2)} S_{t,k}(\gamma) = 0.$$

From these three propositions one finds the following partial differential equations on the symmetric functions of γ:

(1) For any polynomial one variable L and any symmetric polynomial of $n-2$ variables G the equation holds:

$$\sum_{i=1}^{n-1} \frac{1}{\prod_{j\neq i}(\gamma_i - \gamma_j)} D_t(L)(\gamma_i) G(\gamma_1, \ldots \widehat{\gamma_i} \ldots, \gamma_{n-1}) - \sum_{l,m=1}^{n-1} \partial_l \partial_m \times \left[\sum_{i=1}^{n-1} \frac{1}{\prod_{j\neq i}(\gamma_i - \gamma_j)} \gamma_i^{n-1-l} \int_0^{\gamma_i} L(\gamma)\gamma^{n-1-m} \,\mathrm{d}\gamma \, G(\gamma_1, \ldots \widehat{\gamma_i} \ldots, \gamma_{n-1}) \right] = 0. \tag{11}$$

(2) For any symmetric polynomial of $n-3$ variables G the equation holds:

$$\begin{aligned}\mathcal{C}(G) \equiv{}& \sum_{i<j} \frac{1}{(\gamma_i - \gamma_j) \prod_{l\neq i,j}(\gamma_i - \gamma_l)(\gamma_j - \gamma_l)} C_t(\gamma_i, \gamma_j) G(\gamma_1, \ldots \widehat{\gamma_i} \ldots \widehat{\gamma_j} \ldots, \gamma_{n-1}) \\ &- \sum_{l,m=1}^{n-1} \partial_l \partial_m \Bigg[\sum_{i<j} \frac{1}{(\gamma_i - \gamma_j) \prod_{l\neq i,j}(\gamma_i - \gamma_l)(\gamma_j - \gamma_l)} \\ &\times G(\gamma_1, \ldots \widehat{\gamma_i} \ldots \widehat{\gamma_j} \ldots, \gamma_{n-1}) \\ &\left(\gamma_i^{n-l-1} \int_0^{\gamma_i} \frac{\gamma^{n-1-m} - \gamma_j^{n-1-m}}{\gamma - \gamma_j} \,\mathrm{d}\gamma - \gamma_j^{n-l-1} \int_0^{\gamma_j} \frac{\gamma^{n-1-m} - \gamma_i^{n-1-m}}{\gamma - \gamma_i} \,\mathrm{d}\gamma \right) \Bigg] \\ ={}& 0.\end{aligned} \tag{12}$$

(3) For any symmetric polynomial of $n-2$ variables G the equation holds:

$$\mathcal{Q}(G) \equiv \sum_{l=1}^{n-1} \partial_l \left[\sum_{i=1}^{n-1} \frac{1}{\prod_{j\neq i}(\gamma_i - \gamma_j)} \gamma_i^{n-l-1} G(\gamma_1, \ldots \widehat{\gamma_i} \ldots, \gamma_{n-1}) \right] = 0. \tag{13}$$

One must add to the equations (11)–(13) their trivial consequences: i.e. the equations obtained from them by applying an arbitrary number of ∂_i. We claim that in this way all the equations of motion can be described.

Let us illustrate this point considering the simple example $n = 2$. In that case genus is equal to 1 and we have only one variable γ. The equations (12) and (13) are trivial, so we are left with (11). Let us take $L(\gamma) = \gamma^p$, $p = 0, 1, \ldots$. Then (11) turns into

$$p\gamma^{p-1}P(\gamma) + \frac{1}{2}\gamma^p \frac{\mathrm{d}}{\mathrm{d}\gamma}P(\gamma) = \partial_1^2\left(\frac{1}{p+1}\gamma^p\right) \tag{14}$$

and we have only one time τ_1 in that case. As has been said earlier the function γ is the Weierstrass $\mathcal{P}$-function. The equation (14) coincides with the usual equation (5) when $p = 0$. For other p we get the equations on degrees of γ which can be verified for the Weierstrass function. Recall that we were considering the space of functions of the kind

$$\partial_1^k F(t_1, t_2, \gamma). \tag{15}$$

Let us calculate the character of this space modulo the equations (14) and their trivial consequences (those obtained by application of ∂_1). Obviously, using (14) we can express any function of the kind (15) as a linear combination of

$$F_0(t_1, t_2) \qquad F_1(t_1, t_2)\partial_1^m\gamma \qquad F_2(t_1, t_2)\partial_1^m\gamma^2.$$

So, the character is

$$\frac{1}{[2]!}\left(1 + \frac{q}{[1]} + \frac{q^2}{[1]}\right) = \frac{1}{[2]!}(1+q^2)$$

which coincides with (3).

Let us consider the general case. We have the space of symmetric functions of $\gamma_1, \ldots, \gamma_{n-1}$ with coefficients in $t_1, \ldots, t_n$ on which the derivatives $\partial_1, \ldots, \partial_{n-1}$ act. We can also consider the derivatives as coefficients, identifying this space with H_{n-1} which is the space of symmetric polynomials of $\gamma_1, \ldots, \gamma_{n-1}$ with coefficients in $t_1, \ldots, t_n$ and $\partial_1, \ldots, \partial_{n-1}$ (recall that ∂_i and t_j commute). Substracting the exact forms (11) we finish with the space $\widehat{H}_{n-1}$ which is the subspace of H_{n-1} defined by the condition that the degree of the polynomials in every γ_j does not exceed n. We define the spaces $\widehat{H}_j$ ($j \leqslant n-1$) of symmetric polynomials of j variables γ_i whose degree in every variable does not exceed $2n - j - 1$ with coefficients in $t_1, \ldots, t_n$ and $\partial_1, \ldots, \partial_{n-1}$. The action of the operators $\mathcal{C}$ and $\mathcal{Q}$ defined by (12) and (13) can be obviously extended to the action from H_{n-3} to H_{n-1} and from H_{n-2} to H_{n-1} respectively. It is also clear that the images of respectively $\widehat{H}_{n-3}$ and $\widehat{H}_{n-2}$ belong to $\widehat{H}_{n-1}$. One can easily generalize the definitions of $\mathcal{C}$ and $\mathcal{Q}$ allowing them to act respectively from $\widehat{H}_{j-2}$ to $\widehat{H}_j$ and from $\widehat{H}_{j-1}$ to $\widehat{H}_j$. For these operators one has

$$[\mathcal{C}, \mathcal{Q}] = 0 \qquad \mathcal{Q}^2 = 0 \qquad \mathrm{Ker}|_{\widehat{H}_{j-2}\to\widehat{H}_j}(\mathcal{C}) = 0. \tag{16}$$

Now noticing that $\deg(\mathcal{C}) = 2$ and $\deg(\mathcal{Q}) = 1$ one evaluates the character:

$$\begin{aligned}
\frac{1}{[n-1]![n]!}&\left\{\left(\begin{bmatrix}2n-1\\ n-1\end{bmatrix} - q\begin{bmatrix}2n-1\\ n-2\end{bmatrix} + q^2\begin{bmatrix}2n-1\\ n-3\end{bmatrix} - \cdots + (-q)^{n-1}\begin{bmatrix}2n-1\\ 0\end{bmatrix}\right)\right.\\
&-q^2\left(\begin{bmatrix}2n-1\\ n-3\end{bmatrix} - q\begin{bmatrix}2n-1\\ n-4\end{bmatrix}\right.\\
&\left.\left.+\ q^2\begin{bmatrix}2n-1\\ n-5\end{bmatrix} - \cdots + (-q)^{n-3}\begin{bmatrix}2n-1\\ 0\end{bmatrix}\right)\right\}\\
&= \frac{1}{[n]![n-1]!}\left(\begin{bmatrix}2n-1\\ n-1\end{bmatrix} - q\begin{bmatrix}2n-1\\ n-2\end{bmatrix}\right)
\end{aligned}$$

which coincides with the character (3). There is an obvious similarity between what we have done and paper [2].

3. Quantum Toda chain

Following Sklyanin [5] we shall use the same notations for the quantum analogues of classical objects that have been used in the classical case. The Hamiltonian of the quantum periodical Toda chain is given by (1) with p_j, q_j being the canonical operators

$$[p_i, q_j] = \mathrm{i}\hbar\delta_{i,j}.$$

Consider the same definition of L-operator and monodromy matrix as in classics. The monodromy matrix satisfies the relations

$$R(\lambda-\mu)(M(\lambda)\otimes M(\mu)) = (M(\mu)\otimes I)(M(\lambda)\otimes I)R(\lambda-\mu) \tag{17}$$

with $R(\lambda)$ being the quantum R-matrix:

$$R(\lambda) = \lambda - \mathrm{i}\hbar P.$$

The trace of the monodromy matrix provides n commutative integrals of motion.The center of the algebra is created by the quantum determinant

$$A(\lambda)D(\lambda+\mathrm{i}\hbar) - B(\lambda)C(\lambda+\mathrm{i}\hbar) = 1. \tag{18}$$

The idea of using the separated variables in quantum case goes back to [11]. It was developed as a universal method by Sklyanin. Let us briefly review the method of separation of variables following [5]. From the relations (17) one finds, in particular, that

$$[B(\lambda), B(\mu)] = 0.$$

So, presenting the operator $B(\lambda)$ in the form

$$B(\lambda) = b\prod_{j=1}^{n-1}(\lambda-\gamma_j)$$

one gets a family of commuting operators:

$$[b,\gamma_j] = 0 \qquad [\gamma_i,\gamma_j] = 0.$$

As in the classical case the operators b and $a_1 = P$ commute with everything except between themselves:

$$[a_1, b] = \mathrm{i}\hbar b.$$

The idea of the method of separation of variables is in considering the model in b, γ-representation. The canonically conjugated operator to b exists already: this is a_1. The canonically conjugated operators for γ_j are constructed as follows. Consider the operators

$$A(\lambda) = \lambda^n + \lambda^{n-1}a_1 + \cdots + a_n$$
$$D(\lambda) = \lambda^{n-2}d_2 + \cdots + d_n.$$

Then it can be shown that the operators

$$\begin{aligned}\tilde\Lambda_j &= \gamma_j^n + \gamma_j^{n-1}a_1 + \cdots + a_n\\ \Lambda_j &= \gamma_j^{n-2}d_2 + \cdots + d_n\end{aligned} \tag{19}$$

satisfy

$$\tilde\Lambda_j\Lambda_j = 1 \qquad [\Lambda_j, \gamma_g] = \mathrm{i}\hbar\delta_{j,k}\Lambda_j. \tag{20}$$

The order of multipliers in (19) is important.

It is possible to reconstruct the operators $A(\lambda)$, $B(\lambda)$, $C(\lambda)$, $D(\lambda)$ using $a_1, b, \gamma_j, \Lambda_j$. In particular,

$$T(\lambda) = \sum_{k=1}^{n-1}\prod_{j\neq k}\left(\frac{\lambda-\gamma_j}{\gamma_k-\gamma_j}\right)(\Lambda_k+\Lambda_k^{-1}) + \prod_{j\neq 1}^{n-1}(\lambda-\gamma_j)\left(\lambda+a_1+\sum\gamma_j\right). \tag{21}$$

The Hilbert space splits into a direct sum of orthogonal subspaces H_p corresponding to different eigenvalues p of the zero-mode a_1. Let us consider the space H_0, the solution in other subspaces H_p are obtained from the one for H_0 by simple transformation. In H_0 the eigenfunctions of $T(\lambda)$ with the eigenvalue $t(\lambda)$ in γ-representation can be looked for in the form

$$\langle t|\gamma\rangle = \prod_{j=1}^{n-1} Q(\gamma_j).$$

Applying (21) one finds with the following equation for $Q(\gamma)$:

$$t(\gamma)Q(\gamma) = Q(\gamma+\mathrm{i}\hbar) + Q(\gamma-\mathrm{i}\hbar) \tag{22}$$

where $t(\lambda)$ is the eigenvalue of $T(\lambda)$ on $|t\rangle$. In the subspace H_0 this eigenvalue must be a polynomial of the kind

$$t(\lambda) = \lambda^n + \mathrm{O}(\lambda^{n-2}).$$

Equation (22) is an equation with two unknowns, t and Q, so at first glance it is rather useless. However, assuming certain analytical properties of $Q(\gamma)$ this single equation actually defines the spectrum. Namely, require that the function $Q(\gamma)$ is an entire function of γ with infinitely many real zeros. Moreover, impose the following asymptotic:

$$\begin{aligned} &Q(\lambda) \sim \cos\left(\frac{\lambda n}{\hbar}\log\left(\frac{\lambda}{e}\right)+\frac{\pi}{4}\right) && \lambda\sim\infty \\ &Q(\lambda) \sim \mathrm{e}^{\frac{\pi}{\hbar}\lambda n}\cos\left(\frac{\lambda n}{\hbar}\log\left(-\frac{\lambda}{e}\right)+\frac{\pi}{4}\right) && \lambda\sim-\infty. \end{aligned} \tag{23}$$

The normalization of Q is not the same as in [5]. For $n = 4k$ the function Q differs from the function φ from [5] by the multiplier $\exp(\pi\lambda n/2\hbar)$ which is in this case a quasiconstant i.e. it does not disturb equations (22). If $n \neq 4k$ the formulae of [5] require certain corrections which are provided by equations (23). Our normalization basically coincides with the one accepted in [12]. According to [12] it provides the only way to have correct quasiclassical limit. We discuss this limit in the next section.

The main conjecture of the paper [5] is that the spectrum of the model is defined by all the solutions to equations (22) with the analytical properties of t and Q described above and the asymptotic behaviour of Q given by (23). This conjecture was proven by Gaudin and Pasquier [12].

Now we want to discuss the properties of the matrix elements of the operators. To consider the matrix elements we need to know the scalar product in the space of functions of γ_j. This scalar product was found by Sklyanin [5]. We repeat the essential steps because again there will be a minor difference if $n \neq 4k$. Consider an operator $\mathcal{O}$ which is given by a symmetric function $F(\gamma_1, \ldots, \gamma_{n-1})$. The wavefunctions are real, so the matrix element is given by

$$\langle t|\mathcal{O}|t'\rangle = \int_{-\infty}^{\infty}\mathrm{d}\gamma_1\ldots\int_{-\infty}^{\infty}\mathrm{d}\gamma_{n-1}\prod_{j=1}^{n-1}Q(\gamma_j)Q'(\gamma_j)F(\gamma_1,\ldots,\gamma_{n-1})w(\gamma_1,\ldots\gamma_{n-1})$$

where w is a certain weight. Requiring that the operator $T(\lambda)$ for real λ is self-adjoint and using the formula (21) one concludes that

$$w(\gamma_1, \ldots, \gamma_{n-1}) = \prod_{i<j} (\gamma_i - \gamma_j) \tilde{w}(\gamma_1, \ldots, \gamma_{n-1})$$

where $\tilde{w}(\gamma_1, \ldots, \gamma_{n-1})$ is the $\mathrm{i}\hbar$-periodic entire function of its arguments. There are two formal reasons for choosing particular $\tilde{w}$. First, everything under the integral is symmetric with respect to $\gamma_1, \ldots, \gamma_{n-1}$ except for the multiplier $\prod_{i<j}(\gamma_i - \gamma_j)$ in w, so it does not make sense to put in $\tilde{w}$ anything that can be killed by antisymmetrization. Secondly, requiring convergence of the integrals one finds from the asymptotic (23) that $\tilde{w}$ can contain $\exp(-2\pi\gamma_j m/\hbar)$ with $1 \leqslant m \leqslant n-1$. These two requirements fix $\tilde{w}$ up to antisymmetrization:

$$\tilde{w}(\gamma_1, \ldots, \gamma_{n-1}) = \prod_{j=1}^{n-1} \mathrm{e}^{\frac{2\pi}{\hbar}\gamma_j(j-n)}. \tag{24}$$

Again, if $n = 4k$ this essentially coincides with [5]; otherwise there is a minor discrepancy.

An informal reason for taking $\tilde{w}$ in the form of (24) refers to the quasiclassics, and is discussed in the next section.

Similarly to the classical case the algebra of observables $\mathcal{A}_0$ is defined as the algebra generated by all the coefficients of $A(\lambda)$, $\tilde{B}(\lambda) = b^{-1}B(\lambda)$, $\tilde{C}(\lambda) = bC(\lambda)$, $D(\lambda)$. The commutation relations (17) and the quantum determinant (18) provide sufficiently many relations to show that the quantum algebra of observables has the same size as the classical one. To make this statement mathematically rigorous one says that their characters coincide.

Now we formulate an analogue of the conjecture 1 of the previous section.

Conjecture 2. Every quantum observable $\mathcal{O}$ can be presented in the form

$$\mathcal{O} = G_L(t_1, \ldots, t_n) F(b_1, \ldots, b_{n-1}) G_R(t_1, \ldots, t_n)$$

where G_L, F, G_R are polynomials.

This conjecture looks more natural than its classical counterpart and explains the mystery of the latter. The point is that there is no closed formula for the commutation relations of $T(\lambda)$ and $B(\mu)$ which would allow ordering of the operators t_i and b_j.

Taking the matrix element between two eigenstates of the Hamiltonians one can essentially neglect the polynomials G_L and G_R because acting on the eigenstates they produce the eigenvalues. Hence we are mostly interested in the matrix elements of the operators $\mathcal{O}_0 = F(b_1, \ldots, b_{n-1})$ which are given by the integrals of the kind

$$\begin{aligned}\langle t|\mathcal{O}_0|t'\rangle = \int_{-\infty}^{\infty} \mathrm{d}\gamma_1 \ldots \int_{-\infty}^{\infty} \mathrm{d}\gamma_{n-1} \prod_{j=1}^{n-1} Q(\gamma_j) Q'(\gamma_j) \\ \times \prod_{i<j}(\gamma_i - \gamma_j) \prod_{j=1}^{n-1} \tilde{F}(\gamma_1, \ldots, \gamma_{n-1}) \prod_{j=1}^{n-1} \mathrm{e}^{\frac{2\pi}{\hbar}\gamma_j(j-n)}\end{aligned} \tag{25}$$

where

$$\tilde{F}(\gamma_1, \ldots, \gamma_{n-1}) = F(b_1(\gamma), \ldots, b_{n-1}(\gamma)). \tag{26}$$

The function

$$\prod_{i<j}(\gamma_i - \gamma_j) \prod_{j=1}^{n-1} \tilde{f}(\gamma_1, \ldots, \gamma_{n-1})$$

is an antisymmetric polynomial of $\gamma_1, \ldots, \gamma_{n-1}$. Every antisymmetric polynomial can be presented as a linear combination of Schur functions, so, without loss of generality we can replace (26) by

$$\det(F_i(\gamma_j))$$

for some polynomials of one variable $F_1, \ldots, F_{n-1}$. Then the matrix element can be presented as a determinant of an $(n-1) \times (n-1)$-matrix composed of one-fold integrals:

$$\det\left(\int_{-\infty}^{\infty} Q(\gamma)Q'(\gamma)F_i(\gamma)\mathrm{e}^{\frac{2\pi}{\hbar}\gamma(j-n)}\right)\mathrm{d}\gamma. \tag{27}$$

We consider the integral

$$\int_{-\infty}^{\infty} Q(\gamma)Q'(\gamma)F(\gamma)\mathrm{e}^{-\frac{2\pi}{\hbar}\gamma k}\,\mathrm{d}\gamma \tag{28}$$

as a deformation of hyperelliptic integral. The study of the properties of these integrals is very important for understanding the matrix elements of operators in the model.

Since the quantum algebra of observables has the same character as the classical one the equations (11)–(13) must have their quantum counterparts. Hence there must be identities for the integrals (28) from which these quantum counterparts follow. These identities can indeed be found, and we shall describe them in the same order as in the classical case.

Let us introduce the operation Δ which attaches to every function $F(\gamma)$ the function

$$\Delta(F)(\gamma) = F(\gamma + \mathrm{i}\hbar) - F(\gamma - \mathrm{i}\hbar).$$

The operation Δ^{-1} is not always defined, but on polynomials it is. For any polynomial L one can define a polynomial F such that $F(\gamma) = \Delta^{-1}(L)(\gamma)$. For uniqueness we require also that $\Delta^{-1}(L)(0) = 0$. Consider the integral (28) with $Q(\gamma)$, $Q'(\gamma)$ satisfying equations (22) with the eigenvalues $t(\gamma)$ and $t'(\gamma)$. For any given polynomial $L(\lambda)$ construct the polynomial

$$\begin{aligned}(L)_{t,t'}(\gamma) = {} & t(\gamma)\Delta^{-1}(Lt)(\gamma) + t'(\gamma)\Delta^{-1}(Lt')(\gamma) - t(\gamma)\Delta^{-1}(Lt')(\gamma - \mathrm{i}\hbar) \\ & - t'(\gamma)\Delta^{-1}(Lt)(\gamma - \mathrm{i}\hbar) - L(\gamma)t(\gamma)t'(\gamma) + L(\gamma + \mathrm{i}\hbar) - L(\gamma - \mathrm{i}\hbar).\end{aligned} \tag{29}$$

Then we have the following analogue of the propositions 1 and 1′.

Proposition 1″. For any $1 \leqslant k \leqslant n-1$ the following integral vanishes:

$$\int_{-\infty}^{\infty} Q(\gamma)Q'(\gamma)D(L)_{t,t'}(\gamma)\mathrm{e}^{-\frac{2\pi}{\hbar}\gamma k}\,\mathrm{d}\gamma = 0. \tag{30}$$

Using these q-exact forms (29) we can always reduce the degree of the polynomial $F(\gamma)$ under the integral (28) in order that it does not exceed $2n-2$. So, we are left with a finite number of different integrals (28) with $F(\gamma) = \gamma^j$, $j = 0, 1, \ldots, 2n-2$. These integrals are subject to further relations.

In order to define a quantum analogue of the polynomial $C_t(\gamma_1, \gamma_2)$ (10), we need some preparation. Consider the function

$$U(\gamma, \delta) = \frac{t(\gamma) - t(\delta)}{\gamma - \delta}.$$

The notation $\Delta^{-1}(U(\cdot, \delta))(\gamma)$ means that Δ^{-1} is applied to the first argument, i.e.

$$\Delta(\Delta^{-1}(U(\cdot, \delta)))(\gamma) = U(\gamma, \delta).$$

The function U' is defined in the same way replacing t by t'.

Consider the antisymmetric polynomial of two variables

$$C_{t,t'}(\gamma_1,\gamma_2)=R_{t,t'}(\gamma_1,\gamma_2)-R_{t,t'}(\gamma_2,\gamma_1)$$

where $R_{t,t'}(\gamma_1,\gamma_2)$ is defined as follows:

$$\begin{aligned}R_{t,t'}(\gamma_1,\gamma_2)&=t(\gamma_1)\Delta^{-1}(U(\cdot,\gamma_2))(\gamma_1)+t'(\gamma_1)\Delta^{-1}(U'(\cdot,\gamma_2))(\gamma_1)\\&\quad-t(\gamma_1)\Delta^{-1}(U'(\cdot,\gamma_2))(\gamma_1-\mathrm{i}\hbar)-t'(\gamma_1)\Delta^{-1}(U(\cdot,\gamma_2))(\gamma_1-\mathrm{i}\hbar)\\&\quad-\frac{1}{2}\frac{1}{\gamma_1-\gamma_2}(t(\gamma_1)-t(\gamma_2))(t'(\gamma_1)-t'(\gamma_2)).\end{aligned}$$

We also have the following.

Proposition 2″. For any $1<k,l<n-1$ the following integral vanishes:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}Q(\gamma_1)Q'(\gamma_1)Q(\gamma_2)Q'(\gamma_2)C_{t,t'}(\gamma_1,\gamma_2)\mathrm{e}^{-\frac{2\pi}{\hbar}\gamma_1k}\mathrm{e}^{-\frac{2\pi}{\hbar}\gamma_2l}\,\mathrm{d}\gamma_1\,\mathrm{d}\gamma_2=0. \tag{31}$$

Finally, let us define

$$S_{t,t'}(\gamma)=t(\gamma)-t'(\gamma).$$

Then we have the following.

Proposition 3″. For any $1<k<n-1$ the following integral vanishes:

$$\int_{-\infty}^{\infty}Q(\gamma)Q'(\gamma)S_{t,t'}(\gamma)\mathrm{e}^{-\frac{2\pi}{\hbar}\gamma k}\,\mathrm{d}\gamma=0. \tag{32}$$

This is in fact the simplest relation, we consider it last for historical reasons. One obvious consequence of this relation; is orthogonality of the wavefunction.

Let us say a few words about the proof of all these relations. Consider the simplest one (32). We have

$$\begin{aligned}Q(\gamma)Q'(\gamma)S_{t,t'}(\gamma)&=Q(\gamma)Q'(\gamma)(t(\gamma)-t'(\gamma))=Q(\gamma+\mathrm{i}\hbar)Q'(\gamma)+Q(\gamma-\mathrm{i}\hbar)Q'(\gamma)\\&\quad-Q(\gamma)Q'(\gamma+\mathrm{i}\hbar)-Q(\gamma)Q'(\gamma-\mathrm{i}\hbar).\end{aligned}$$

So,

$$\begin{aligned}\int_{-\infty}^{\infty}Q(\gamma)Q'(\gamma)S_{t,t'}(\gamma)\mathrm{e}^{-\frac{2\pi}{\hbar}\gamma k}\,\mathrm{d}\gamma&=\int_{-\infty}^{\infty}Q(\gamma+\mathrm{i}\hbar)Q'(\gamma)\mathrm{e}^{-\frac{2\pi}{\hbar}\gamma k}\,\mathrm{d}\gamma\\&\quad+\int_{-\infty}^{\infty}Q(\gamma-\mathrm{i}\hbar)Q'(\gamma)\mathrm{e}^{-\frac{2\pi}{\hbar}\gamma k}\,\mathrm{d}\gamma-\int_{-\infty}^{\infty}Q(\gamma)Q'(\gamma+\mathrm{i}\hbar)\mathrm{e}^{-\frac{2\pi}{\hbar}\gamma k}\,\mathrm{d}\gamma\\&\quad-\int_{-\infty}^{\infty}Q(\gamma)Q'(\gamma-\mathrm{i}\hbar)\mathrm{e}^{-\frac{2\pi}{\hbar}\gamma k}\,\mathrm{d}\gamma.\end{aligned}$$

By shift of contour the first integral cancels the fourth one and the second integral cancels the third one. The shift of contour is possible because the function Q behaves asymptotically on the line $\mathrm{Im}(\gamma)=$constant, essentially in the same way as it behaves at the real axis. The relations (30) and (31) are proven similarly, but that requires more sophisticated calculations.

Using equations (30)–(32) we can write down equations similar to (11)–(13) which would guarantee that the number of operators (character of the algebra of observables) is correct. The corresponding calculation does not differ from the one presented earlier for the classical case.

4. Quasiclassical case

We had a number of identities described by propositions 1, 1′, 1″; 2, 2′, 2″; 3′, 3″. They appear to be reflections of the same structure. The goal of this section is to explain the relation between the different levels of deformation.

Consider the quasiclassical quantization of the Toda chain. We had the following formulae for the symplectic form and corresponding 1-form in the coordinates t and γ:

$$\begin{aligned} \alpha &= \sum_{j=1}^{n-1} \log \Lambda(\gamma_j)\, \mathrm{d}\gamma_j \\ \omega &= \mathrm{d}\alpha = \sum_{j=1} \sum_{k=2}^{n} \frac{\gamma_j^{n-k}}{\sqrt{P(\gamma_j)}}\, \mathrm{d}t_k \wedge \mathrm{d}\gamma_j . \end{aligned} \tag{33}$$

In this section we shall ignore the contribution from the centre of mass variables a_1 and b which is easy to find if needed. Using these formulae one immediately writes a quasiclassical proposal for the wavefunction of the states with eigenvalues t in γ-representation:

$$\Psi_t(\gamma) = \mu^{\frac{1}{2}} \exp\left(\frac{1}{\mathrm{i}\hbar}\int^{\gamma} \alpha\right) = \prod_{i<j}(\gamma_i-\gamma_j)^{\frac{1}{2}} \prod_{j=1}^{n-1}\left(\frac{1}{P(\gamma_j)}\right)^{\frac{1}{4}} \exp\left(\frac{1}{\mathrm{i}\hbar}\int^{\gamma_j} \log\Lambda(\gamma)\,\mathrm{d}\gamma\right) \tag{34}$$

where μ is defined as follows

$$\wedge^{n-1}(\omega) = \mu\, \mathrm{d}\gamma_1 \wedge \ldots \wedge \mathrm{d}\gamma_{n-1} \wedge \mathrm{d}t_2 \wedge \ldots \wedge \mathrm{d}t_n .$$

The relation of the formula (34) to the exact quantum formulae that we had before is clear.

(1) The multiplier $\prod_{i<j}(\gamma_i-\gamma_j)^{\frac{1}{2}}$ is a piece of the weight of integration w (25). Another one will come from the second wavefunction in the matrix elements. The wavefunction without this multiplier will be denoted by $\tilde{\Psi}_t(\gamma)$.

(2) The function $\tilde{\Psi}_t(\gamma)$ splits into a product of the expressions

$$\left(\frac{1}{P(\gamma_j)}\right)^{\frac{1}{4}} \exp\left(\frac{1}{\mathrm{i}\hbar}\int^{\gamma_j} \log\Lambda(\gamma)\,\mathrm{d}\gamma\right)$$

which has to be related to the quasiclassical limit of the function Q. One should be careful here because the function $\log\Lambda(\gamma)$ is multivalued, so, the branches must be defined. Moreover, we would probably need linear combination of the wavefunctions corresponding to different branches.

Recall that

$$\Lambda(\gamma) = \tfrac{1}{2}\left(T(\gamma) + \sqrt{T^2(\gamma)-4}\right)$$

where $T(\lambda)$ is a polynomial of degree n. The polynomial $P(\gamma) = T^2(\gamma) - 4$ has $2n$ real simple roots $\lambda_1 < \cdots < \lambda_{2n}$. We define the function $\Lambda(\gamma)$ on the plane with the cuts along the intervals; $I_0 = (-\infty, \lambda_1]$, $I_1 = [\lambda_2, \lambda_3], \ldots, I_n = [\lambda_{2n}, \infty)$. Requiring that $\log\Lambda(\gamma \pm \mathrm{i}0)$ are real for $\gamma > \lambda_{2n}$ then, obviously,

$$\log\Lambda(\gamma+\mathrm{i}0) + \log\Lambda(\gamma-\mathrm{i}0) = 0 \qquad \gamma \in I_n .$$

Continuing analytically $\log\Lambda(\gamma)$ into the plane with the cuts one finds that

$$\begin{aligned} \log\bar{\Lambda}(\gamma) = \log\Lambda(\bar{\gamma}) \qquad \text{and} \qquad & \log\Lambda(\gamma+\mathrm{i}0) + \log\Lambda(\gamma-\mathrm{i}0) \\ = 2\pi\mathrm{i}(n-j) \qquad & \gamma \in I_j . \end{aligned}$$

The variable γ_j belongs classically to I_j. So, we need the wavefunction real on the entire real axis and not containing the factors $\exp(-\frac{1}{\hbar}s)$ with s real when all the variables are in the classically permited places. This requires, first, taking different brances of $\log\Lambda$ for different γ_j and, second, taking a sum of two wavefunctions with $\gamma_j+\mathrm{i}0$ and $\gamma_j-\mathrm{i}0$ for every γ_j. The result is

$$\tilde{\Psi}_t(\gamma)=\prod \mathrm{e}^{\frac{\pi}{\hbar}(j-n)\gamma_j}Q_{\mathrm{qc}}(\gamma_j)$$

where

$$Q_{\mathrm{qc}}(\gamma)=V(\gamma+\mathrm{i}0)+V(\gamma-\mathrm{i}0)\qquad V(\gamma)=\left(-\frac{1}{P(\gamma)}\right)^{\frac{1}{4}}\exp\left(\frac{1}{\mathrm{i}\hbar}\int^{\gamma}\log\Lambda(\gamma)\,\mathrm{d}\gamma\right).$$

The branch of $\log\Lambda$ is defined above; $(-\frac{1}{P(\gamma)})^{\frac{1}{4}}$ is real positive for $\lambda_{2j-1}<\gamma<\lambda_{2j}$.

To ensure that $Q_{\mathrm{qc}}(\gamma)$ is a single-valued function in the plane with the cuts the Bohr–Sommerfeld quantization condition must hold

$$J_j=\int_{a_j}\log\Lambda(\gamma)\,\mathrm{d}\gamma=\pi\hbar(2n_j+1). \tag{35}$$

The integrals J_j are the classical actions. The cuts of Q_{qc} come from condensation of zeros of the quantum Q in the quasiclassical limit.

Let us consider now the quasiclassical limit $\hbar\to 0$ of the matrix elements. This limit makes sense literally if two condition are satisfied:

(1) The quantum numbers are large. In our case it means that the zones $[\lambda_{2j},\lambda_{2j+1}]$ do not collapse.

(2) The states $|t\rangle$ and $|t'\rangle$ are close, i.e. the eigenvalues of the Hamiltonians are close: $t'_j-t_j=\mathrm{O}(\hbar)$.

Consider the matrix element (25) for such close states. It consists of the integrals

$$\int_{-\infty}^{\infty}Q(\gamma)Q'(\gamma)F_i(\gamma)\mathrm{e}^{\frac{2\pi}{\hbar}(j-n)\gamma}\,\mathrm{d}\gamma.$$

From the quasiclassical estimation of Q, Q' one concludes that

$$\begin{aligned}&\int_{-\infty}^{\infty}Q(\gamma)Q'(\gamma)F_i(\gamma)\mathrm{e}^{\frac{2\pi}{\hbar}(j-n)\gamma}\,\mathrm{d}\gamma\\&\qquad\xrightarrow{\hbar\to 0}4\int_{I_j}\frac{1}{\sqrt{P(\gamma)}}\cos\left(\frac{1}{\hbar}\int^{\gamma}\mathrm{Re}\,(\log\Lambda(\gamma'+\mathrm{i}0))\,\mathrm{d}\gamma'+\frac{\pi}{4}\right)\\&\qquad\times\cos\left(\frac{1}{\hbar}\int^{\gamma}\mathrm{Re}\,(\log\Lambda'(\gamma'+\mathrm{i}0))\,\mathrm{d}\gamma'+\frac{\pi}{4}\right)F_i(\gamma)\,\mathrm{d}\gamma.\end{aligned}$$

We have

$$\begin{aligned}&2\cos\left(\frac{1}{\hbar}\int^{\gamma}\mathrm{Re}\,(\log\Lambda(\gamma'+\mathrm{i}0))\,\mathrm{d}\gamma'+\frac{\pi}{4}\right)\cos\left(\frac{1}{\hbar}\int^{\gamma}\mathrm{Re}\,(\log\Lambda'(\gamma'+\mathrm{i}0))\,\mathrm{d}\gamma'+\frac{\pi}{4}\right)\\&\qquad=\cos\left(\frac{1}{\hbar}\int^{\gamma}\mathrm{Re}\,(\log\Lambda'(\gamma'+\mathrm{i}0)-\log\Lambda(\gamma'+\mathrm{i}0))\,\mathrm{d}\gamma'\right)\\&\qquad\quad-\sin\left(\frac{1}{\hbar}\int^{\gamma}\mathrm{Re}\,(\log\Lambda'(\gamma'+\mathrm{i}0)+\log\Lambda(\gamma'+\mathrm{i}0))\,\mathrm{d}\gamma'\right).\end{aligned}$$

The second term in the r.h.s. can be thrown away because it is rapidly oscillating in the classical limit. The variation $\log\Lambda(\gamma)-\log\Lambda'(\gamma)$ is estimated as follows. Consider the classical solution with $T_{\mathrm{cl}}(\lambda)=t(\lambda)$ where $t(\lambda)$ is one of the eigenvalues. This is the place where using the same notations for classical and quantum observables can be misleading, so we explicitly mark the classical ones. The eigenvalue $t'(\lambda) = T_{\mathrm{cl}}(\lambda) + \delta T(\lambda)$. One has (in further calculations we neglect contributions of order $\mathrm{o}(\hbar)$):

$$\delta \log \Lambda(\gamma) \equiv \log \Lambda'(\gamma) - \log \Lambda(\gamma) = \frac{\delta T(\gamma)}{\sqrt{P(\gamma)}}.$$

How to find $\delta T(\gamma)$? The quasiclassical states are subject to the Bohr–Sommerfeld quantization conditions (35). The quantum numbers n_j are quasiclassically large: $n_j = \mathrm{O}(\hbar^{-1})$, but their differences for the close states are finite $k_j \equiv n'_j - n_j = \mathrm{O}(1)$. Hence

$$\delta \int_{a_j} \log \Lambda(\gamma)\,\mathrm{d}\gamma = \int_{a_j} \frac{\delta T(\gamma)}{\sqrt{P(\gamma)}}\,\mathrm{d}\gamma = \sum_{l=1}^{n-1} \delta t_{n-l+1} \int_{a_j} \frac{\gamma^{l-1}}{\sqrt{P(\gamma)}}\,\mathrm{d}\gamma = \hbar k_j. \tag{36}$$

The matrix

$$A^{-1}_{lj} = \int_{a_j} \frac{\gamma^{l-1}}{\sqrt{P(\gamma)}}\,\mathrm{d}\gamma$$

is the inverse for the matrix used in the definition of the normalized Abelian differentials ω_j. Hence

$$\delta t_{n-l+1} = \hbar k_j A_{jl}$$

which means that

$$\delta \log \Lambda(\gamma)\,\mathrm{d}\gamma = \hbar \sum k_j \omega_j.$$

Thus the quasiclassical matrix element for the close states is

$$\langle t|\mathcal{O}|t'\rangle = \int_{I_1} \frac{\mathrm{d}\gamma_1}{\sqrt{P(\gamma_1)}} \cdots \int_{I_{n-1}} \frac{\mathrm{d}\gamma_1}{\sqrt{P(\gamma_{n-1})}} \prod_{i<j}(\gamma_i - \gamma_j) F(\gamma_1, \ldots, \gamma_{n-1}) \prod_j 2\cos(\Phi_k(\gamma_j)). \tag{37}$$

Recall the notation $\Phi_k(\gamma) = \int^{\gamma} k_j \omega_j$. This is the same expression as for the Fourier coefficient in (7). At first glance there are two disagreements: in (7) we integrate over a_j, and we have $\exp(\mathrm{i}\Phi_k(\gamma_j))$ under the integral. Actually, these two disagreements compensate each other because $a_j = (I_j + \mathrm{i}0) - (I_j - \mathrm{i}0)$ and $\Phi_k(\gamma + \mathrm{i}0) + \Phi_k(\gamma - \mathrm{i}0) = 0$ when $\gamma \in I_j$. Notice that the states $|t\rangle$ are not normalized.

It is not a surprise that we have found the Fourier coefficient as the quasiclassical limit of the matrix element. We have performed all the calculations in order to have the complete mathematical picture. On the other hand from the point of view of physics one can argue as follows.

Consider the action-angle variables $J_1, \ldots, J_{n-1}$, $\theta_1, \ldots, \theta_j$. The Bohr–Sommerfeld quantization in this variables does not give the correct quantum result, but is still correct quasiclassically. Consider the eigenstate of Hamiltonians $|t\rangle$ and the eigenstates of the angles $|\theta\rangle$. Quasiclassically one has for the wavefunction:

$$\langle t|\theta\rangle = \frac{1}{\sqrt{\prod J_k}} \exp\left(\frac{1}{\mathrm{i}\hbar} \sum J_k \theta_k\right). \tag{38}$$

Consider an operator $\mathcal{O}$. This operator can be, at least quasiclassically, ordered in such a way that J's are to the left of θ's. Then the classical shape of the corresponding observable on the solution with given values of integrals (t) is

$$\mathcal{O}_{\mathrm{cl}}(\theta_1, \ldots, \theta_{n-1}) = \lim_{\hbar \to 0} \frac{\langle t|\mathcal{O}|\theta\rangle}{\langle t|\theta\rangle}.$$

Insert the complete set of eigenstates into this formula

$$\mathcal{O}_{\mathrm{cl}}(\theta_1,\ldots,\theta_{n-1}) = \lim_{\hbar\to 0}\frac{\langle t|\mathcal{O}|\theta\rangle}{\langle t|\theta\rangle} = \sum_{t'}\frac{\langle t|\mathcal{O}|t'\rangle}{\langle t'|t'\rangle}\frac{\langle t'|\theta\rangle}{\langle t|\theta\rangle}. \tag{39}$$

Quasiclassically only close states are important, for which we have from (38):

$$\frac{\langle t'|\theta\rangle}{\langle t|\theta\rangle} = \mathrm{e}^{-\mathrm{i}\Sigma k_j\theta_j}.$$

Now it is obvious that (39) gives the Fourier transformation of $\mathcal{O}_{\mathrm{cl}}(\theta_1,\ldots,\theta_{n-1})$. It is clear from equation (37) that quasiclassically

$$\langle t'|t'\rangle = \langle t|t\rangle + \mathrm{O}(\hbar) \qquad \langle t|t\rangle = \det(A).$$

So there is complete agreement with formula (7).

Let us consider two close states introducing the notation: $S_{t,k}(\gamma) = t'(\gamma) - t(\gamma)$. The quantum matrix element goes to the classical Fourier coefficient when $\hbar \to 0$. If we consider the classical Fourier coefficient with $k = 0$ it describes the classical limit of the quantum expectation value $\langle t|\mathcal{O}|t\rangle$. So, it is no surprise that we have the following sequences:

$$\begin{aligned} &D_{t,t'}(L)(\gamma)\xrightarrow{\hbar\to 0}D_{t,k}(L)(\gamma)\xrightarrow{k=0}D_t(L)(\gamma)\\ &C_{t,t'}(\gamma_1,\gamma_2)\xrightarrow{\hbar\to 0}C_{t,k}(\gamma_1,\gamma_2)\xrightarrow{k=0}C_t(\gamma_1,\gamma_2)\\ &S_{t,t'}(\gamma)\xrightarrow{\hbar\to 0}S_{t,k}(\gamma)\xrightarrow{k=0}0. \end{aligned} \tag{40}$$

Thus, there are two levels of deformation of the hyperelliptic differentials. The impression is that the quantum deformation is very natural, and that the classical mechanics appears as a strange intermediate case.

5. Conclusions

The identities (30)–(32) present the main result of this paper. They show that the matrix elements of the arbitrary operator in the quantum Toda chain can be expressed with the help of finitely many integrals which possess remarkable properties.

There is a difference between what we have and the sine-Gordon theory in infinite volume. Indeed, the matrix elements for the Toda chain are given by integrals of arbitrary deformed differentials with respect to a fixed half-basis of deformed cycles. In the sine-Gordon case we took an arbitrary half-basis. The reason for that is the difference in the type of reality conditions.

In this connection it is very important to consider the deformation of the Toda chain with a trigonometric R-matrix and more complicated reality conditions, which is much closer to the sine-Gordon case. We hope that in this situation there would be a complete duality between deformed differentials and deformed cycles.

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